SOLUTION OF THE GENERAL EQUATION OF THE FIFTH DEGREE.

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§. 8.

From the 5 unknown quantities (y, v, z, u, t) of an equation of the 5th degree:

I. $$x^5 + bx^3 - cx^2 + dx - e = 0$$

$(2\times3\times4\times5)=120$ solution-sums can be formed, of which, nevertheless, every five are fifth roots of the same dignant, e. g.: $(y+v'+z''+u'''+t'''')^5 = (y'+v''+z'''+u''''+t)^5 = (y''+v'''+z''''+u+t')^5 = (y'''+v''''+z+u'+t'')^5 = (y''''+v+z'+u''+t''')^5$.

The equation for determining the solution-sums of the 5th degree will therefore have 120 dimensions; nevertheless, since 5 of its unknown quantities are roots ofthe same dignant, it will in strict significance be an equation of the 24th degree.

This equation can be found, that is to say, its coefficients can be given in rational functions of those of equation I.

I construct from equation I. one whose unknown quantities are the 5 th powers of y, v, z, u and t; thus

II. $$x^{25} - Ax^{20} + Bx^{15} - Cx^{10} + Dx^5 - E = 0,$$

wherein: $A = 5(-bc + e)$

$$B = (b^5 - 5b^2d + 5b^2c^2 - 15bce + 5bd^2 + 5c^2d + 10e^2)$$

$$C = (-5b^3de + 5b^2c^2e + 5b^2cd^2 - 5bc^3d - 15bce^2 + 10bd^2e + c^5 + 10c^2de - 5cd^3 + 10e^3)$$

$$D = (-5bce^3 + 5bd^2e^2 + 5c^2de^2 - 5cd^3e + d^5 + 5e^4)$$

$$E = e^5,$$

and, considering that in this case $a = 0$ and properly

$$A = (a^5 - 5a^3b + 5a^2c + 5ab^2 - 5ad - 5bc + 5e),$$

thus from these 6 equations the value of a will be given by a function of A, B, C, D, E, and, indeed, by means of an equation of the 125th degree; which, nevertheless, in strict significance (for x^5) will be only an equation of the 25th degree.

The value of a^5 is known to me by equation I. (here $a = 0$), and if I now divide the found equation of the 25th degree by $x^5 - a^5$, there remains an equation of the 24th degree for x^5. The $5\times24 = 120$ values of this x are indeed the 120 solution-sums; for, if I consider equation II. in its

origin, its unknown quantities are $y, y', y'', y''', y'''', v, v', v'', v''', v''''$, etc.; a thus always denotes the coefficient of one dimension in an equation which has y, v, z, u, t as factors of its summands, and has each one of them combined with $\sqrt[5]{1}$ in such a manner that the product of all these summands will be equal to e. This condition the solution-sums alone fulfil.

Every solution-sum must therefore be represented by means of *one* irrational quantity, which stands under a radical of the 5th degree; this irrational quantity can admit of no more and no less than 120 significations (I call these its harmonic values), since, if I assume one of the solution-sums $= \sqrt[5]{F}$, the remaining 119 possible solution-sums are easily formed by changing the signs of the radical quantities occurring in this value.

It follows from this also that 24 of these solution-sums, namely, 24 which are not roots of the same dignant and whose value I group by combining the expression $\sqrt[5]{F}$ according to what is outside in the form $\sqrt[5]{F}$ (while in the radicals of this dignant F remains rational), must receive the determination of their values by changes of sign in the radicals of this dignant.

These radicals can therefore be of such form only that 24 harmonic values are given by them; that is to say, only such roots can occur whose index is a factor of 24, and, particularly, none of these radical quantities (part of the dignant F) can have the index 5.

§. 9.

Among those 24 solution-sums whose element y is combined with the value $\sqrt[5]{1} = 1$, there cannot be two or more which are 5th roots of the same dignant.

I chose this circumstance as characteristic of those solution-sums which are related as roots of the same dignant.

The 24 solution-sums in which the summand y occurs without the imaginary factor $\sqrt[5]{1}$, thus represent 24 exactly determined groups of solution-sums.

It is found always that among these 24 sums there are 4 which stand in such a relation that, added to a and divided by 5, they express the unknown quantity y (§. 7, V.).

Consequently in the 24 groups already described 6 series of 4 such groups are formed.

As characteristic of these 6 series I assume that two of their elements have the same form; thus, for example, $(y+v')$ occurs in all six.

Thus, for the 24 solution-sums with the element y, which are not related as roots of the same dignant, I have constructed the 6 following series or groups of the second class:

$$\begin{array}{llll}
\text{III.} & (1) & \begin{array}{l}(y+v'+z''+u'''+t'''')\\(y+v''+z''''+u'+t''')\\(y+v'''+z'+u''''+t'')\\(y+v''''+z'''+u''+t')\end{array} & (4)\ \begin{array}{l}(y+v'+z''+u''''+t''')\\(y+v''+z''''+u'''+t')\\(y+v'''+z'+u''+t'''')\\(y+v''''+z'''+u'+t'')\end{array}\\[2ex]
 & (2) & \begin{array}{l}(y+v'+z'''+u''''+t'')\\(y+v''+z'+u'''+t'''')\\(y+v'''+z''''+u''+t')\\(y+v''''+z''+u'+t''')\end{array} & (5)\ \begin{array}{l}(y+v'+z'''+u''+t'''')\\(y+v''+z'+u''''+t''')\\(y+v'''+z''''+u'+t'')\\(y+v''''+z''+u'''+t')\end{array}\\[2ex]
 & (3) & \begin{array}{l}(y+v'+z''''+u''+t''')\\(y+v''+z'''+u''''+t')\\(y+v'''+z''+u'+t'''')\\(y+v''''+z'+u'''+t')\end{array} & (6)\ \begin{array}{l}(y+v'+z''''+u'''+t'')\\(y+v''+z'''+u'+t'''')\\(y+v'''+z''+u''''+t')\\(y+v''''+z'+u''+t''')\end{array}
\end{array}$$

From the 6 characteristic sums:

$$\begin{array}{llll}
\text{IV.} & (1)\ (y+v'+z''+u'''+t'''') & (4)\ (y+v'+z''+u''''+t''')\\
 & (2)\ (y+v'+z'''+u''''+t'') & (5)\ (y+v'+z'''+u''+t'''')\\
 & (3)\ (y+v'+z''''+u''+t''') & (6)\ (y+v'+z''''+u'''+t'')
\end{array}$$

I further construct groups of the 3d and 4th classes, by threes and by twos, as this is signified by their juxtaposition and superposition.

While I sought and found external marks of the first group arranged according to mutual fitness and of the second group (§. 7) constructed likewise upon internal grounds, I am now permitted to consider these marks as authorized for application respectively to the following groupings.

I therefore mention as the mark of the third group-formation, that the characteristic equations involved (IV. 1 to 6) have in common $(y+v'+z'')$, $(y+v'+z''')$, or $(y+v'+z'''')$.

Consequently these equations group themselves by twos and by threes: the latter, if it is assumed that only such elements of the 6 equations in question belong to this same group by threes, which have no equal element but y and v'.

With this arrangement the operation is at once reduced to the finding of the possible permutations of y, v, z, u and t, since in all functions of y, v, z, u and t which enter rationally (especially in the symmetrical functions and their symmetrical parts) the possible permutations of the 5 unknown quantities are found again.

§. 10.

I assume that one of the 24 solution-sums, for example

I. $\sqrt[5]{F(1)} = f(1) = (y+v'+z''+u'''+t'''')$,

may be so represented that, by the introduction of the true values of $\sqrt[5]{1}$, I would have (for the meaning of p and q, see §. 2.)

$$\text{II.}\quad f(1) = y+\tfrac{1}{4}v(-1+\sqrt{5}+p+q)+\tfrac{1}{4}z(-1-\sqrt{5}-p+q)$$
$$+\tfrac{1}{4}u(-1-\sqrt{5}+p-q)+\tfrac{1}{4}t(-1+\sqrt{5}-p-q)$$
$$= y-\tfrac{1}{4}(v+z+u+t)+\tfrac{1}{4}[(v+t)-(z+u)]\sqrt{5}+\tfrac{1}{4}[(v+u)-(z+t)\,p$$
$$+\tfrac{1}{4}[(v+z)-(u+t)q$$
$$= \tfrac{5}{4}y+\tfrac{1}{4}[(v+t)-(z+u)]\sqrt{5}+\tfrac{1}{4}[(v+u)-(z+t)]p+\tfrac{1}{4}[(v+z)-(u+t)]q.$$

If I now employ the auxiliaries:

$(v+t)-(z+u) = a_{,}$, $(v+u)-(z+t) = b_{,}$, $(v+z)-(u+t) = d_{,}$, I find:

$$\begin{array}{ll}
\text{III.}\; f(1) = \tfrac{1}{4}(5y+a_{,}\sqrt{5}+b_{,}p+d_{,}q) & f(13) = \tfrac{1}{4}(5y+a_{,}\sqrt{5}+d_{,}p+b_{,}q) \\
f(2) = \tfrac{1}{4}(5y-a_{,}\sqrt{5}-d_{,}p+b_{,}q) & f(14) = \tfrac{1}{4}(5y-a_{,}\sqrt{5}-b_{,}p+d_{,}q) \\
f(3) = \tfrac{1}{4}(5y-a_{,}\sqrt{5}+d_{,}p-b_{,}q) & f(15) = \tfrac{1}{4}(5y-a_{,}\sqrt{5}+b_{,}p-d_{,}q) \\
f(4) = \tfrac{1}{4}(5y+a_{,}\sqrt{5}-b_{,}p-d_{,}q) & f(16) = \tfrac{1}{4}(5y+a_{,}\sqrt{5}-d_{,}p-b_{,}q) \\
f(5) = \tfrac{1}{4}(5y+b_{,}\sqrt{5}+d_{,}p+a_{,}q) & f(17) = \tfrac{1}{4}(5y+b_{,}\sqrt{5}+a_{,}p+d_{,}q) \\
f(6) = \tfrac{1}{4}(5y-b_{,}\sqrt{5}-a_{,}p+d_{,}q) & f(18) = \tfrac{1}{4}(5y-b_{,}\sqrt{5}-d_{,}p+a_{,}q) \\
f(7) = \tfrac{1}{4}(5y-b_{,}\sqrt{5}+a_{,}p-d_{,}q) & f(19) = \tfrac{1}{4}(5y-b_{,}\sqrt{5}+d_{,}p-a_{,}q) \\
f(8) = \tfrac{1}{4}(5y+b_{,}\sqrt{5}-d_{,}p-a_{,}q) & f(20) = \tfrac{1}{4}(5y+b_{,}\sqrt{5}-a_{,}p-d_{,}q) \\
f(9) = \tfrac{1}{4}(5y+d_{,}\sqrt{5}+a_{,}p+b_{,}q) & f(21) = \tfrac{1}{4}(5y+d_{,}\sqrt{5}+b_{,}p+a_{,}q) \\
f(10) = \tfrac{1}{4}(5y-d_{,}\sqrt{5}-b_{,}p+a_{,}q) & f(22) = \tfrac{1}{4}(5y-d_{,}\sqrt{5}-a_{,}p+b_{,}q) \\
f(11) = \tfrac{1}{4}(5y-d_{,}\sqrt{5}+b_{,}p-a_{,}q) & f(23) = \tfrac{1}{4}(5y-d_{,}\sqrt{5}+a_{,}p-b_{,}q) \\
f(12) = \tfrac{1}{4}(5y+d_{,}\sqrt{5}-a_{,}p-b_{,}q) & f(24) = \tfrac{1}{4}(5y+d_{,}\sqrt{5}-b_{,}p-a_{,}q)
\end{array}$$

I multiply together these expressions by twos, and, indeed, those of the same group in which the same unknown quantities are combined with those values of $\sqrt{5}$ whose product $= +5$; thus, for example:

$$f(1)f(4) = \frac{(5y+a_{,}\sqrt{5})^2-(b_{,}p-d_{,}q)^2}{16}$$

$$f(1)\;(4) = \frac{5(5y^2+a_{,}^2+b_{,}^2+d_{,}^2) + 2\sqrt{5}(5a_{,}y+b_{,}d_{,}-b_{,}^2+d_{,}^2)}{16}$$

or, since $(5y^2+a_{,}^2+b_{,}^2+d_{,}^2) = [4y^2+(-v-z-u-t)^2+3v^2+3z^2+3u^2+3t^2$

$$-2(vz+vu+vt+zu+zt+ut)]$$

$$= 4(y^2+v^2+z^2+u^2+t^2) = -8b.$$

IV.

$$
\begin{aligned}
f(1\;)(\;4) &= \tfrac{1}{8}[-20b+\sqrt{5}(5a_{,}y+b_{,}d_{,}-b_{,}^{2}+d_{,}^{2})]\\
f(2\;)(\;3) &= \tfrac{1}{8}[-20b-\sqrt{5}(5a_{,}y+b_{,}d_{,}-b_{,}^{2}+d_{,}^{2})]\\
f(5\;)(\;8) &= \tfrac{1}{8}[-20b+\sqrt{5}(5b_{,}y+a_{,}d_{,}-d_{,}^{2}+a_{,}^{2})]\\
f(6\;)(\;7) &= \tfrac{1}{8}[-20b-\sqrt{5}(5b_{,}y+a_{,}d_{,}-d_{,}^{2}+a_{,}^{2})]\\
f(9\;)(12) &= \tfrac{1}{8}[-20b+\sqrt{5}(5d_{,}y+a_{,}b_{,}-a_{,}^{2}+b_{,}^{2})]\\
f(10)(11) &= \tfrac{1}{8}[-20b-\sqrt{5}(5d_{,}y+a_{,}b_{,}-a_{,}^{2}+b_{,}^{2})]\\
f(13)(16) &= \tfrac{1}{8}[-20b+\sqrt{5}(5b_{,}y+a_{,}d_{,}-a_{,}^{2}+d_{,}^{2})]\\
f(14)(15) &= \tfrac{1}{8}[-20b-\sqrt{5}(5b_{,}y+a_{,}d_{,}-a_{,}^{2}+d_{,}^{2})]\\
f(17)(20) &= \tfrac{1}{8}[-20b+\sqrt{5}(5a_{,}y+b_{,}d_{,}-d_{,}^{2}+b_{,}^{2})]\\
f(18)(19) &= \tfrac{1}{8}[-20b-\sqrt{5}(5a_{,}y+b_{,}d_{,}-d_{,}^{2}+b_{,}^{2})]\\
f(21)(24) &= \tfrac{1}{8}[-20b+\sqrt{5}(5d_{,}y+a_{,}b_{,}-b_{,}^{2}+a_{,}^{2})]\\
f(22)(23) &= \tfrac{1}{8}[-20b-\sqrt{5}(5d_{,}y+a_{,}b_{,}-b_{,}^{2}+a_{,}^{2})]
\end{aligned}
$$

Now, since $5a_{,}y = 4a_{,}y+a_{,}(-v-z-u-t)$:

$$
\begin{aligned}
(\;5a_{,}y+b_{,}d_{,}) &= 4(yv+yt+zu-yz-yu-vt)\\
(-b_{,}^{2}\;+d_{,}^{2}\;) &= 4(\qquad vz+ut-vu-zt)\\
(\;5b_{,}y+a_{,}d_{,}) &= 4(yv+yu+zt-yz-yt-vu)\\
(-a_{,}^{2}\;+d_{,}^{2}\;) &= 4(\qquad vt+zu-vz-ut)\\
(\;5d_{,}y+a_{,}b_{,}) &= 4(yv+yz+ut-yu-yt-vz)\\
(-a_{,}^{2}\;+b_{,}^{2}\;) &= 4(\qquad vu+zt-vt-zu)
\end{aligned}
$$

whence there results:

V.

$$
\begin{aligned}
(5a_{,}y+b_{,}d_{,}-b_{,}^{2}+d_{,}^{2}) &= 4[(yv+yt+vz+zu+ut)-(yz+yu+vu+vt+zt)]\\
(5a_{,}y+b_{,}d_{,}+b_{,}^{2}-d_{,}^{2}) &= 4[(yv+yt+vu+zu+zt)-(yz+yu+vz+vt+ut)]\\
(5b_{,}y+a_{,}d_{,}-a_{,}^{2}+d_{,}^{2}) &= 4[(yv+yu+vt+zu+zt)-(yz+yt+vz+vu+ut)]\\
(5b_{,}y+a_{,}d_{,}+a_{,}^{2}-d_{,}^{2}) &= 4[(yv+yu+vz+ut+zu)-(yz+yt+vu+vt+zu)]\\
(5d_{,}y+a_{,}b_{,}-a_{,}^{2}+b_{,}^{2}) &= 4[(yv+yz+vu+zt+ut)-(yu+yt+vz+vt+zu)]\\
(5d_{,}y+a_{,}b_{,}+a_{,}^{2}-b_{,}^{2}) &= 4[(yv+yz+vt+zu+ut)-(yu+yt+vz+vu+zt)]
\end{aligned}
$$

From IV. and V.:

VI. $f(1)(4) = \tfrac{1}{2}[-5b+[(yv+yt+vz+zu+ut)-(yz+yu+vu+vt+zt)]\sqrt{5}]$

$f(2)(3) = \tfrac{1}{2}[-5b-[(yv+yt+vz+zu+ut)-(yz+yu+vu+vt+zt)]\sqrt{5}]$

with easy analogies for the products: $f(5)(8)$, etc. (V.)

From VI. there further results:

VII. $f(1)(2)(3)(4) = \tfrac{1}{4}[25b^{2}-5[(yv+yt+vz+zu+ut)-(yz+yu+vu+vt+zt)]^{2}]$,

or, since $[(yv+yt+vz+zu+ut)-(yz+yu+vu+vt+zt)]^2$

$$=[(y^2v^2\ldots 10t.)+2d+2(y^2vt+y^2zu+v^2yz+v^2ut+z^2yt+z^2vu+u^2yv+u^2zt+t^2yu+t^2vz)-2(y^2vz+y^2vu+y^2zt+y^2ut+v^2yu+v^2zt+v^2yt+v^2zu+z^2yv+z^2vt+z^2yu+z^2ut+u^2yz+u^2vt+u^2yt+u^2vz+t^2yz+t^2vu+t^2yv+t^2zu)]$$

$$=(b^2+12d)+4(y^2vt+y^2zu+v^2yz+v^2ut+z^2yt+z^2vu+u^2yv+u^2zt+t^2yu+t^2vz),$$

there follows: $f(1)(2)(3)(4)=\frac{1}{4}[25b^2-5(b^2+12d)-20(y^2vt\ldots 10t.\text{ as above}),$
or

VIII. (1) $f(\ 1)(\ 2)(\ 3)(\ 4)=5[b^2-3d-(y^2vt+y^2zu+v^2yz+v^2ut+z^2yt+z^2vu+u^2yv+u^2zt+t^2yu+t^2vz)],$

and analogously:

(2) $f(\ 5)(\ 6)(\ 7)(\ 8)=5[b^2-3d-(y^2vu+y^2zt+v^2yt+v^2zu+z^2yv+z^2ut+u^2yz+u^2vt+t^2yu+t^2vz)],$

(3) $f(\ 9)(10)(11)(12)=5[b^2-3d-(y^2vz+y^2ut+v^2yu+v^2zt+z^2yt+z^2vu+u^2yz+u^2vt+t^2yv+t^2zu)],$

(4) $f(13)(14)(15)(16)=5[b^2-3d-(y^2vu+y^2zt+v^2yz+v^2ut+z^2yu+z^2vt+u^2yt+u^2vz+t^2yv+t^2zu)],$

(5) $f(17)(18)(19)(20)=5[b^2-3d-(y^2vt+y^2zu+v^2yu+v^2zt+z^2yv+z^2ut+u^2yt+u^2vz+t^2yz+t^2vu)],$

(6) $f(21)(22)(23)(24)=5[b^2-3d-(y^2vz+y^2ut+v^2yt+v^2zu+z^2yu+z^2vt+u^2yv+u^2zt+t^2yz+t^2vu)].$

I put the expression occurring in V.

$$(yv+yt+vz+zu+ut)=m,^{*}\text{ thus}$$
$$(yz+yu+vu+vt+zt)=b-m;$$

hence there follows from VII.:

IX. $$f(1)(2)(3)(4)=\frac{25b^2-5(2m-b)^2}{4}=5(b^2-m^2+mb).$$

(*To be continued.*)

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*I will always use the symbol m with this signification only.